Third Monthly Progress Report

on

Thermal Strain Analysis

of

Advanced Manned-Spacecraft Heat Shields

NASA Contract NAS 9-1986
Period 26 October 1963 to 29 November 1963

ARA Report
#30

Prepared by:

Daniel H. Platus
Daniel H. Platus, Project Scientist

Shoichi Uchiyama
Schoichi Uchiyama, Project Scientist

Approved by:

B. Mazelsky
B. Mazelsky, President
ARA, Inc.

OTS PRICE

XEROX $ 1.00 fs.
MICROFILM $ 0.50 mf.

Third Monthly Progress Report on Thermal Strain Analysis of Advanced Manned-Spacecraft Heat Shields

NASA Contract NAS 9-1986

As a result of the technical progress made during the first two monthly periods and in view of certain problems encountered during the third monthly period, it became necessary to revise the original program schedule. It was agreed at a joint meeting of key ARA and AGC personnel, that the original program schedule would be modified to include additional study phases believed most pertinent to the overall objective of the program. Consequently, other phases had to be changed, accordingly, to maintain the original workload. The revised program schedule is shown in Fig. 1 and reference will be made to this schedule in reporting progress made during the third monthly period.

Phase A - Deriviation of Basic Equations

This phase was extended to include an investigation of the singularity which arises on the axis-of-symmetry, in the non-axisymmetric case. Results of this study, which complete Phase A of the program, are summarized in Appendix 1 of this report. It was concluded that points on the axis-of-symmetry could be considered in the non-axisymmetric case but that the additional programming required and the complication introduced would probably not justify the improved accuracy to be gained.

Phase A" - Investigation of Engineering Models

The heat shield, as described in NASA Contract NAS 9-1986, consists of a sandwich shell structure to which an ablative material is bonded. The overall thickness of the composite structure requires a three-dimensional or "thick-shell" analysis. However, certain regions of the structure consist of extremely thin layers which, by themselves, can satisfy the usual thin-shell criteria. The face plates of the sandwich substructure, for example, are specified as [illegible] comparison with 2 in. for the honeycomb core. In view of their high elastic modulus the face plates constitute a substantial fraction of the total stifness of the structure and, consequently, cannot be ignored. Another region of extreme importance is the bond line where the ablation is joined to the metal substructure. Although the layer is only approximately 0.030 inches thick compared with thicknesses of the order of [illegible] inch for the ablator, the stress distribution throughout the bond is of utmost importance in considering the over-all structural integrity of the heat shield. Considerable complication results if these thin layers are treated by three-dimensional theory owing to the large differences in the grid spacings between directions normal [illegible] in plane with the thin layers, in conjunction with finite difference solutions to the partial differential equations. Consequently, it becomes necessary to treat these thin layers in a manner which avoids further subdivision of these layers into thinner layers. This can be accomplished by treating each layer by thin shell theory, which requires only a two-dimensional solution of the displacement-equilibrium equations at the median surface of the shell. The stress and strain distributions throughout the shell thickness are then obtained using the Kirchhoff bending hypothesis for thin shells. The method is summarized in Appendix 2 for a flat plate using Cartesian coordinates. This effort constitutes 25% of the total effort for this phase. During the next [illegible] the method will be extended to spherical and toroidal curvilinear coordinates of interest in the heat shield analysis.

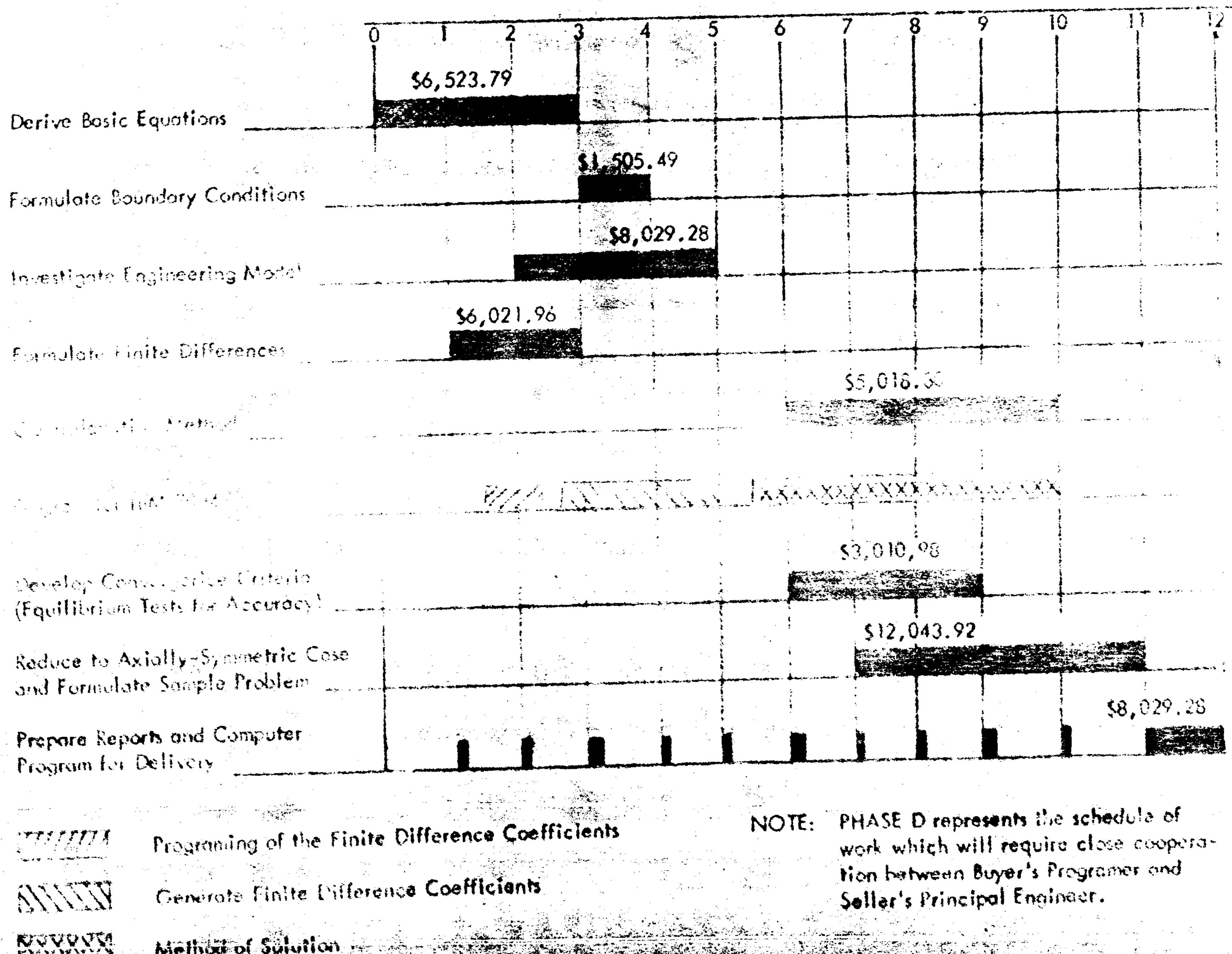
<u>Phase G. - Report Preparation</u>

The work contained herein represents [illegible] of the effort and brings to 20% the effort expended to date in the report writing phase of the study.

FIGURE 1

THERMAL STRAIN ANALYSIS OF ADVANCED MANNED SPACECRAFT VEHICLES

PROGRAM SCHEDULE



NOTE: PHASE D represents the schedule of work which will require close cooperation between Buyer's Programer and Seller's Principal Engineer.

Appendix I

Singularities at an Axis $\rho = 0$

Singularities at the Axis $\varphi = 0$

The equilibrium equations in spherical coordinates in terms of displacements are written in the form

$$(\lambda+2\mu)\frac{\partial^2 u}{\partial R^2} + \frac{\mu}{R^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\mu}{R^2\sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} - \frac{2(\lambda+2\mu)}{R}\frac{\partial u}{\partial R} + \frac{\mu\cos\varphi}{R^2\sin\varphi}\frac{\partial u}{\partial \varphi}$$

$$- \frac{2(\lambda+2\mu)}{R^2}u + \frac{\lambda+\mu}{R}\frac{\partial^2 v}{\partial R\partial\varphi} + \frac{(\lambda+\mu)\cos\varphi}{R\sin\varphi}\frac{\partial v}{\partial R} - \frac{\lambda+3\mu}{R^2}\frac{\partial v}{\partial \varphi} \tag{1}$$

$$- \frac{(\lambda+3\mu)\cos\varphi}{R^2\sin\varphi}v + \frac{\lambda+\mu}{R\sin\varphi}\frac{\partial^2 w}{\partial R\partial\theta} - \frac{\lambda+3\mu}{R^2\sin\varphi}\frac{\partial w}{\partial \theta} = \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{11}}}\frac{\partial T}{\partial R}$$

$$\frac{\lambda+\mu}{R}\frac{\partial^2 u}{\partial R\partial\varphi} + \frac{2(\lambda+2\mu)}{R^2}\frac{\partial u}{\partial \varphi} + \mu\frac{\partial^2 v}{\partial R^2} + \frac{\lambda+2\mu}{R^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{\mu}{R^2\sin^2\varphi}\frac{\partial^2 v}{\partial \theta^2}$$

$$+ \frac{2\mu}{R}\frac{\partial v}{\partial R} + \frac{(\lambda+2\mu)\cos\varphi}{R^2\sin\varphi}\frac{\partial v}{\partial \varphi} - \frac{\lambda+2\mu}{R^2\sin^2\varphi}v + \frac{\lambda+\mu}{R^2\sin\varphi}\frac{\partial^2 w}{\partial\theta\partial\varphi} \tag{2}$$

$$- \frac{(\lambda+3\mu)\cos\varphi}{R^2\sin^2\varphi}\frac{\partial w}{\partial \theta} = \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{22}}}\frac{\partial T}{\partial \varphi}$$

$$\frac{\lambda+\mu}{R\sin\varphi}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{2(\lambda+2\mu)}{R^2\sin\varphi}\frac{\partial u}{\partial \theta} + \frac{\lambda+\mu}{R^2\sin\varphi}\frac{\partial^2 v}{\partial\theta\partial\varphi} + \frac{(\lambda+3\mu)\cos\varphi}{R^2\sin^2\varphi}\frac{\partial v}{\partial \theta}$$

$$+ \mu\frac{\partial^2 w}{\partial R^2} + \frac{\mu}{R^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\lambda+2\mu}{R^2\sin^2\varphi}\frac{\partial^2 w}{\partial \theta^2} + \frac{2\mu}{R}\frac{\partial w}{\partial R} + \frac{\mu\cos\varphi}{R^2\sin\varphi}\frac{\partial w}{\partial \varphi} \tag{3}$$

$$- \frac{\mu}{R^2\sin^2\varphi}w = \frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{33}}}\frac{\partial T}{\partial \theta}$$

The temperature terms on the right hand sides of Eqs (1), (2) and (3) are equivalent to body forces defined as

$$\left.\begin{aligned}
\frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{11}}}\frac{\partial T}{\partial R} &= F_R(R,\varphi,\theta)/\sqrt{g_{11}}\\
\frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{22}}}\frac{\partial T}{\partial \varphi} &= F_\varphi(R,\varphi,\theta)/\sqrt{g_{22}}\\
\frac{(3\lambda+2\mu)\alpha(T)}{\sqrt{g_{33}}}\frac{\partial T}{\partial \theta} &= F_\theta(R,\varphi,\theta)/\sqrt{g_{33}}
\end{aligned}\right\} \tag{4}$$

Third Monthly Progress Report on Thermal Strain Analysis
of Advanced Manned-Spacecraft Heat Shields

NASA Contract NAS 9-1986

As a result of the technical progress made during the first two monthly periods and in view of certain problems encountered during the third monthly period, it became necessary to revise the original program schedule. It was agreed at a joint meeting of key ARA and AGC personnel, that the original program schedule would be modified to include additional study phases believed most pertinent to the overall objective of the program. Consequently, other phases had to be changed, accordingly, to maintain the original workload. The revised program schedule is shown in Fig. 1 and reference will be made to this schedule in reporting progress made during the third monthly period.

Phase A - Derivation of Basic Equations

This phase was extended to include an investigation of the singularity which arises on the axis-of-symmetry, in the non-axisymmetric case. Results of this study, which complete Phase A of the program, are summarized in Appendix 1 of this report. It was concluded that points on the axis-of-symmetry could be considered in the non-axisymmetric case but that the additional programming required and the complication introduced would probably not justify the improved accuracy to be gained.

Phase A" - Investigation of Engineering Models

The heat shield, as described in NASA Contract NAS 9-1986, consists of a sandwich shell structure to which an ablative material is bonded. The overall thickness of the composite structure requires a three-dimensional or "thick-shell" analysis. However, certain regions of the structure consist of extremely thin layers which, by themselves, can satisfy the usual thin-shell criteria. The face plates of the sandwich

If the elastic constants are Temperature dependent the equilibrium equations (1), (2) and (3) are written in the form

$$\sum_{i=1}^{10} \left[(A_{ki} + A'_{ki}) U_i + (B_{ki} + B'_{ki}) V_i + (C_{ki} + C'_{ki}) W_i \right] = F_k \quad , (k = R, \varphi, \theta) \tag{5}$$

where

A_{ki}, B_{ki}, C_{ki} = functions of coordinates (R, φ, θ) and elastic constants λ and μ

$A'_{ki}, B'_{ki}, C'_{ki}$ = functions of coordinates (R, φ, θ) and elastic constants $\lambda(T)$ and $\mu(T)$, where T is the heat shield Temperature which is a function of the coordinates (R, φ, θ)

U_i, V_i, W_i = functions of displacements $u(R,\varphi,\theta)$, $v(R,\varphi,\theta)$ and $w(R,\varphi,\theta)$ and their respective derivatives of the first and second orders, respectively.

F_k = body force expressed as $(3\lambda + 2\mu)\,\alpha(T)\,\frac{\partial T}{\partial \alpha_k}$

Eq.(1) may be further shortened into the form

$$\sum_{m=1}^{3} \sum_{i=1}^{10} \left(G_{mki} + G'_{mki} \right) \Phi_{mi} = F_k \quad , (k = R, \varphi, \theta) \tag{6}$$

where

$G_{1,2,3\,ki} = A_{ki}, B_{ki}, C_{ki}$

$G'_{1,2,3\,ki} = A'_{ki}, B'_{ki}, C'_{ki}$

$\Phi_{1,2,3\,i} = U_i, V_i, W_i$

It is first considered that the elastic constants are independent of Temperature. In This case, Eq.(6) becomes, deleting the symbol of summation,

$$G_{mki}(R,\varphi,\theta)\,\Phi_{mi}(R,\varphi,\theta) = F_k(R,\varphi,\theta) \tag{7}$$

Replacing the variables R, φ, and θ of Eq.(7) by $R-R'$, $\varphi-\varphi'$ and $\theta-\theta'$, respectively, and integrating the result with respect to the variables $(R-R', \varphi-\varphi', \theta-\theta')$ over a finite volume V_k gives

$$\iiint_{V_k} G_{mki}\,\Phi_{mi}\,dV' = \iiint_{V_k} F_k\,dV' \tag{8}$$

where $dV' = (R-R')^2 \sin(\varphi-\varphi')\,d(R-R')\,d(\varphi-\varphi')\,d(\theta-\theta')$ and the finite volume V_k is bounded as

$$R_{K1} \le R-R' \le R_{K2}$$

$$\varphi_{K1} \le \varphi-\varphi' \le \varphi_{K2}$$

$$\theta_{K1} \le \theta-\theta' \le \theta_{K2} \; .$$

Let

$$\left.\begin{aligned} I_R &= \int_{R_{K1}}^{R_{K2}} G_{mki}\,(R-R')^2\,d(R-R') \\ I_{R\varphi} &= \int_{\varphi_{K1}}^{\varphi_{K2}} I_R \sin(\varphi-\varphi')\,d(\varphi-\varphi') \\ I_{R\varphi\theta} &= \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi}\,d(\theta-\theta') \end{aligned}\right\} \tag{9}$$

Integration of the function $G_{mki}\Phi_{mi}$ with respect to $(R-R')$ gives

$$\int_{R_{K1}}^{R_{K2}} G_{mki}\,\Phi_{mi}\,(R-R')^2\,d(R-R') = I_R\,\Phi_{mi}\Big|_{R_{K1}}^{R_{K2}} - \int_{R_{K1}}^{R_{K2}} I_R\,\frac{\partial \Phi_{mi}}{\partial (R-R')}\,d(R-R') \; . \tag{10}$$

Integration of Eq.(10) with respect to $(\varphi-\varphi')$, after multiplying both sides by $\sin(\varphi-\varphi')$, gives

$$\int_{\varphi_{K1}}^{\varphi_{K2}} I_R \Phi_{mi}\Big|_{R_{K1}}^{R_{K2}} \sin(\varphi-\varphi')\, d(\varphi-\varphi') - \int_{\varphi_{K1}}^{\varphi_{K2}}\int_{R_{K1}}^{R_{K2}} I_R \frac{\partial \Phi_{mi}}{\partial(R-R')} \sin(\varphi-\varphi')\, d(R-R')\, d(\varphi-\varphi') \tag{11}$$

$$= I_{R\varphi}\Phi_{mi}\Big|_{\substack{\varphi_{K1}\\R_{K1}}}^{\substack{\varphi_{K2}\\R_{K2}}} - \int_{\varphi_{K1}}^{\varphi_{K2}} I_{R\varphi}\frac{\partial \Phi_{mi}}{\partial(\varphi-\varphi')}\, d(\varphi-\varphi') - \int_{\varphi_{K1}}^{\varphi_{K2}}\int_{R_{K1}}^{R_{K2}} I_R \frac{\partial \Phi_{mi}}{\partial(R-R')} \sin(\varphi-\varphi')\, d(R-R')\, d(\varphi-\varphi')$$

Integration of Eq.(11) with respect to $(\theta-\theta')$ gives

$$\iiint G_{mKi}\Phi_{mi}(R-R')^2 \sin(\varphi-\varphi')\, d(R-R')\, d(\varphi-\varphi')\, d(\theta-\theta')$$

$$= I_{R\varphi\theta}\Phi_{mi}\Big|_{\substack{\theta_{K1}\\\varphi_{K1}\\R_{K1}}}^{\substack{\theta_{K2}\\\varphi_{K2}\\R_{K2}}} - \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta}\frac{\partial \Phi_{mi}}{\partial(\theta-\theta')}\, d(\theta-\theta') - \int_{\varphi_{K1}}^{\varphi_{K2}}\int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi}\frac{\partial \Phi_{mi}}{\partial(\varphi-\varphi')}\, d(\varphi-\varphi')\, d(\theta-\theta') \tag{12}$$

$$- \int_{R_{K1}}^{R_{K2}}\int_{\varphi_{K1}}^{\varphi_{K2}}\int_{\theta_{K1}}^{\theta_{K2}} I_R \frac{\partial \Phi_{mi}}{\partial(R-R')} \sin(\varphi-\varphi')\, d(R-R')\, d(\varphi-\varphi')\, d(\theta-\theta')$$

At a point $(R_{Kc}, \varphi_{Kc}, \theta_{Kc})$, Eq.(12) becomes

$$\iiint G_{mKi}\Phi_{mi}(R_{Kc}-R')^2 \sin(\varphi_{Kc}-\varphi')\, d(R_{Kc}-R')\, d(\varphi_{Kc}-\varphi')\, d(\theta_{Kc}-\theta')$$

$$= I_{R\varphi\theta}\Phi_{mi}\Big|_{\substack{\theta_{K1}\\\varphi_{K1}\\R_{K1}}}^{\substack{\theta_{K2}\\\varphi_{K2}\\R_{K2}}} - \int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta}\frac{\partial \Phi_{mi}}{\partial\theta'}\, d\theta' + \int_{\varphi_{K1}}^{\varphi_{K2}}\int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi}\frac{\partial \Phi_{mi}}{\partial\varphi'}\, d\varphi'\, d\theta' \tag{13}$$

$$- \int_{R_{K1}}^{R_{K2}}\int_{\varphi_{K1}}^{\varphi_{K2}}\int_{\theta_{K1}}^{\theta_{K2}} I_R \frac{\partial \Phi_{mi}}{\partial R'} \sin(\varphi_{Kc}-\varphi')\, dR'\, d\varphi'\, d\theta'$$

The right hand side of Eq.(8) becomes

$$-\int_{R_{K1}}^{R_{K2}}\int_{\varphi_{K1}}^{\varphi_{K2}}\int_{\theta_{K1}}^{\theta_{K2}} F_K(R_{Kc}-R', \varphi_{Kc}-\varphi', \theta_{Kc}-\theta')(R_{Kc}-R')^2 \sin(\varphi_{Kc}-\varphi')\, dR'\, d\varphi'\, d\theta' \qquad (13a)$$

By the definition of Kelvin's point force, diminishing the force field V_K' indefinitely always including a point ($R'=0$, $\varphi'=0$, $\theta=0$) gives

$$\left.\begin{aligned}
&\lim_{V_K'\to 0}\int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi\theta}\frac{\partial \Phi_{mi}}{\partial \theta'}\, d\theta' \approx 0\\
&\lim_{V_K'\to 0}\int_{\varphi_{K1}}^{\varphi_{K2}}\int_{\theta_{K1}}^{\theta_{K2}} I_{R\varphi}\frac{\partial \Phi_{mi}}{\partial \varphi'}\, d\varphi'\, d\theta' \approx 0\\
&\lim_{V_K'\to 0}\iiint I_R\frac{\partial \Phi_{mi}}{\partial R'}\sin(\varphi_{Kc}-\varphi')\, dR'\, d\varphi'\, d\theta' \approx 0\\
&\lim_{V_K'\to 0}\iiint F_K(R_{Kc}-R')^2\sin(\varphi_{Kc}-\varphi')\, dR'\, d\varphi'\, d\theta' = {}^{o}F_{Kc}
\end{aligned}\right\} \qquad (14)$$

But

$$\iiint_{V_K} F_K\, dV' = F_K V_K \qquad (15)$$

and, hence,

$${}^{o}F_{Kc} = F_K V_K \qquad (16)$$

Hence from Eqs. (13), (13a), (14) and (16), Eq.(8) becomes

$$\left.\frac{I_{R\varphi\theta}\Phi_{mi}}{V_K}\right|_{\theta_{K1},\varphi_{K1},R_{K1}}^{\theta_{K2},\varphi_{K2},R_{K2}} = F_K\Big|_{R_{Kc},\varphi_{Kc},\theta_{Kc}} \qquad (17)$$

where $\Phi_{mi}\Big|_{\substack{\theta_{K1}\\ \varphi_{K1}\\ R_{K1}}}^{\substack{\theta_{K2}\\ \varphi_{K2}\\ R_{K1}}}$ may be found by taking the average of eight surrounding points.

Appendix ~~2~~ C

Thin-Shell Interface Conditions for Stress Analysis of Thick Laminate Structures

Thin-Shell Interface Conditions for Stress Analysis of Thick Laminate Structures

Equations are derived for treating the stresses in a thick-shell laminate structure in the neighborhood of a thin layer which, by itself, can satisfy the Kirchhoff bending hypothesis for thin shells. It is shown that the thin layer can be treated by an equivalent interface condition which relates the displacements of the median surface of the shell to the discontinuous normal and shear stresses on the adjoining surfaces. From continuity of displacements across the thin layer the interface stresses can be eliminated to yield three simultaneous partial differential equations for the three displacement components at the interface. The analysis is presented for a flat plate using a system of Cartesian coordinates and will be generalized later to the curvilinear coordinate systems of interest in the heat shield analysis.

Consider a thin plate of thickness b with its median surface lying in the parallel x-y plane and the distance z measured from the median surface. The temperature and, consequently, the coefficient of thermal expansion and modulus of elasticity will be allowed to vary through the plate thickness so that the median surface will not, in general, bisect the plate thickness. With this generality, the thin plate itself can consist of a laminate of different materials. According to Kirchhoff's bending hypothesis the strain-displacement relations for a point (x, y, z) in the plate are given by (Ref 1)

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} \\ \epsilon_y &= \frac{\partial v}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\,\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned}\right\} \qquad (1)$$

where u, v, and w are displacements of a point (x, y) on the median surface and ϵ_x, ϵ_y and γ_{xy} are the normal strains and shear strain, respectively in the x-y plane. The stress-strain relations are given by

$$\left.\begin{aligned}
\sigma_x &= \frac{E(x,y,z)}{1-\nu^2}\left[(\epsilon_x + \nu\epsilon_y) - (1+\nu)\,\alpha(x,y,z)\,T(x,y,z)\right] \\
\sigma_y &= \frac{E(x,y,z)}{1-\nu^2}\left[(\epsilon_y + \nu\epsilon_x) - (1+\nu)\,\alpha(x,y,z)\,T(x,y,z)\right] \\
\tau_{xy} &= \frac{E(x,y,z)}{2(1+\nu)}\,\gamma_{xy}
\end{aligned}\right\} \qquad (2)$$

where σ_x and σ_y are normal stresses and τ_{xy} is the shear stress in the x-y plane. The normal stress σ_z and shear stresses τ_{xz} and τ_{yz} are usually small in comparison with the stress components of Eq.(2) and are neglected in thin shell theory. For the problem under consideration, however, the thin shell will be subjected to both normal and shear stresses over its lateral surfaces and it is desired to relate the difference or discontinuity of these stresses across the shell to the displacements of the median surface. These relationships may be obtained from the equations of equilibrium expressed in terms of displacements using the Kirchhoff bending hypothesis of Eq.(1). The equilibrium equations in terms of stresses are given by

$$\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0
\end{aligned}\right\} \qquad (3)$$

Writing the stresses of Eq. (2) in terms of displacements using Eqs. (1) and substituting the results in Eq. (3), the equilibrium equations in terms of displacements become

$$
\left.
\begin{aligned}
&\frac{E(x,y,z)}{1-\nu^2} f_1(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} f_2(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} f_3(x,y) \\
&- \frac{z E(x,y,z)}{1-\nu^2} g_1(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} g_2(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} g_3(x,y) \\
&+ \frac{\partial \tau_{xz}}{\partial z} = \frac{E(x,y,z)}{1-\nu}\frac{\partial}{\partial x}\left[\alpha(x,y,z)\,T(x,y,z)\right] + \frac{1}{1-\nu}\frac{\partial E(x,y,z)}{\partial x}\alpha(x,y,z)\,T(x,y,z) \\
&\frac{E(x,y,z)}{1-\nu^2} f_1'(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} f_2'(x,y) + \frac{1}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} f_3'(x,y) \\
&- \frac{z E(x,y,z)}{1-\nu^2} g_1'(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial x} g_2'(x,y) - \frac{z}{1-\nu^2}\frac{\partial E(x,y,z)}{\partial y} g_3'(x,y) \\
&+ \frac{\partial \tau_{yz}}{\partial z} = \frac{E(x,y,z)}{1-\nu}\frac{\partial}{\partial y}\left[\alpha(x,y,z)\,T(x,y,z)\right] + \frac{1}{1-\nu}\frac{\partial E(x,y,z)}{\partial y}\alpha(x,y,z)\,T(x,y,z) \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0
\end{aligned}
\right\} \quad (4)
$$

where

$$
\left.
\begin{aligned}
f_1(x,y) &= \frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 v}{\partial x\partial y} + \frac{1-\nu}{2}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}\right) \\
f_2(x,y) &= \frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} \\
f_3(x,y) &= \frac{1-\nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
g_1(x,y) &= \frac{\partial^3 w}{\partial x^3} + \nu\frac{\partial^3 w}{\partial x\partial y^2} + (1-\nu)\frac{\partial^3 w}{\partial x\partial y^2} = \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\partial y^2} \\
g_2(x,y) &= \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} \\
g_3(x,y) &= (1-\nu)\frac{\partial^2 w}{\partial x\partial y}
\end{aligned}
\right\} \quad (5)
$$

$$
\left.
\begin{aligned}
f_1'(x,y) &= \frac{\partial^2 v}{\partial y^2} + \nu \frac{\partial^2 u}{\partial x \partial y} + \frac{1-\nu}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \right) \\
f_2'(x,y) &= \frac{1-\nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
f_3'(x,y) &= \frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x} \\
g_1'(x,y) &= \frac{\partial^3 w}{\partial y^3} + \nu \frac{\partial^3 w}{\partial x^2 \partial y} + (1-\nu) \frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y} \\
g_2'(x,y) &= (1-\nu) \frac{\partial^2 w}{\partial x \partial y} \\
g_3'(x,y) &= \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}
\end{aligned}
\right\} \qquad (5)
$$

If the first two of Eqs. (4) are integrated across the plate thickness there results

$$
\left.
\begin{aligned}
& f_1(x,y)\, D_0 + f_2(x,y) \frac{\partial D_0}{\partial x} + f_3(x,y) \frac{\partial D_0}{\partial y} - g_1(x,y)\, D_1 - g_2(x,y) \frac{\partial D_1}{\partial x} \\
& \qquad - g_3(x,y) \frac{\partial D_1}{\partial y} + \tau_{xz}\Big|_2 - \tau_{xz}\Big|_1 = \frac{\partial N_T}{\partial x} \\
& f_1'(x,y)\, D_0 + f_2'(x,y) \frac{\partial D_0}{\partial x} + f_3'(x,y) \frac{\partial D_0}{\partial y} - g_1'(x,y)\, D_1 - g_2'(x,y) \frac{\partial D_1}{\partial x} \\
& \qquad - g_3'(x,y) \frac{\partial D_1}{\partial y} + \tau_{yz}\Big|_2 - \tau_{yz}\Big|_1 = \frac{\partial N_T}{\partial y}
\end{aligned}
\right\} \qquad (6)
$$

where the quantities D_0, D_1 and N_T are defined by

$$
\left.
\begin{aligned}
D_0 &= \frac{1}{1-\nu^2} \int E(x,y,z)\, dz \\
D_1 &= \frac{1}{1-\nu^2} \int z\, E(x,y,z)\, dz \\
N_T &= \frac{1}{1-\nu} \int \alpha(x,y,z)\, E(x,y,z)\, T(x,y,z)\, dz
\end{aligned}
\right\} \qquad (7)
$$

and $\tau_{xz}|_1$, $\tau_{xz}|_2$, etc. are the respective shear stresses on the two surfaces of the plate. If the median surface is determined such that

$$D_1 = 0 \; , \tag{8}$$

which is, in fact, the condition defining the median or "neutral" surface, then Eqs. (6) reduce to two expressions for the shear stress discontinuities across the thin plate in terms of the median surface displacements; i.e.,

$$\left.\begin{aligned} \tau_{xz}|_2 - \tau_{xz}|_1 &= \frac{\partial N_T}{\partial x} - f_1(x,y)\, D_0 - f_2(x,y)\,\frac{\partial D_0}{\partial x} - f_3(x,y)\,\frac{\partial D_0}{\partial y} \\ \tau_{yz}|_2 - \tau_{yz}|_1 &= \frac{\partial N_T}{\partial y} - f_1'(x,y)\, D_0 - f_2'(x,y)\,\frac{\partial D_0}{\partial x} - f_3'(x,y)\,\frac{\partial D_0}{\partial y} \end{aligned}\right\} . \tag{9}$$

A third equation, which is necessary to define the three displacement components u, v and w at the median surface, is obtained from a consideration of equilibrium of forces normal to the plane of the plate. It is shown in Ref. 2 that this expression of equilibrium can be written as

$$\frac{\partial^2 M_x}{\partial x^2} - 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -p - N_x \frac{\partial^2 w}{\partial x^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - N_y \frac{\partial^2 w}{\partial y^2} \; , \tag{10}$$

where p is the lateral pressure loading on the plate and the N's and M's are sectional forces and moments defined by

$$\left.\begin{aligned} N_x &= \int \sigma_x \, dz \; , & N_y &= \int \sigma_y \, dz \; , & N_{xy} &= \int \tau_{xy}\, dz \\ M_x &= \int z\,\sigma_x \, dz \; , & M_y &= \int z\,\sigma_y\, dz \; , & M_{xy} &= -\int z\,\tau_{xy}\, dz \end{aligned}\right\} . \tag{11}$$

Substituting for σ_x, σ_y and τ_{xz} from Eqs.(2), with the definition, Eq.(8), of the median surface, the sectional quantities of Eq.(11) become

$$
\left.
\begin{aligned}
N_x &= D_0\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) - N_T \\
N_y &= D_0\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) - N_T \\
N_{xy} &= \frac{1-\nu}{2} D_0\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\
M_x &= -D_2\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right) - M_T \\
M_y &= -D_2\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) - M_T \\
M_{xy} &= (1-\nu) D_2 \frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\right\}, \qquad (12)
$$

where

$$
\left.
\begin{aligned}
D_2 &= \frac{1}{1-\nu^2}\int z^2 E(x,y,z)\, dz \\
M_T &= \frac{1}{1-\nu}\int z\, \alpha(x,y,z)\, E(x,y,z)\, T(x,y,z)\, dz
\end{aligned}
\right\}. \qquad (13)
$$

The lateral pressure, p, acting on the thin plate is simply the difference between the normal stresses $\sigma_z|_1$ and $\sigma_z|_2$ acting on the two surfaces; i.e.,

$$p = \sigma_z|_2 - \sigma_z|_1 \ . \qquad (14)$$

Hence, on substituting the sectional forces and moments defined by Eqs.(12) in Eq.(10), an expression is obtained analogous to Eqs.(9) for the discontinuity of normal stresses across the thin plate in terms of the three displacement components at the median surface. This equation is found to be .

$$
\begin{aligned}
\sigma_z|_2 - \sigma_z|_1 = {} & \frac{\partial^2}{\partial x^2}\left[D_2\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\right] + 2(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\left(D_2 \frac{\partial^2 w}{\partial x\,\partial y}\right) \\
& + \frac{\partial^2}{\partial y^2}\left[D_2\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)\right] - \frac{\partial^2 w}{\partial x^2}\left[D_0\left(\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right) - N_T\right] \qquad (15) \\
& - 2\frac{\partial^2 w}{\partial x\,\partial y}\left[\frac{1-\nu}{2} D_0\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] - \frac{\partial^2 w}{\partial y^2}\left[D_0\left(\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right) - N_T\right] + \nabla^2 M_T .
\end{aligned}
$$

If it is assumed that the displacements at the surfaces of the two media in contact with the thin layer under consideration are equal to the displacements in this layer at the median surface, then the surface stresses may be expressed in terms of these displacements using Hooke's law with the respective material properties of the two adjoining media. Thus Eqs. (9) and (15) become three partial differential equations in the three displacement components u, v and w at the thin-shell interface. These equations will replace the general three-dimensional equations at the "interface" nodes resulting in only one node at each such interface through the thick laminate structure. Once the three displacement components in the interface plane are determined, the stress distributions throughout the thin layer are obtained from the foregoing thin-shell analysis.

References

1. H. S. Tsien, "Similarity Laws for Stressing Heated Wings", Journal of the Aeronautical Sciences, Vol. 20, No. 1, Jan. 1953

2. S. Timoshenko, "Theory of Plates and Shells", McGraw Hill, 1940, p. 300.